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On the reduction of the multidimensional stationary Schrödinger equation to a first-order equation and its relation to the pseudoanalytic function theory

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Abstract

Given a particular solution of a one-dimensional stationary Schrödinger equation this equation of second order can be reduced to a first-order linear ordinary differential equation. This is done with the aid of an auxiliary Riccati differential equation. In the present work we show that the same fact is true in a multidimensional situation also. For simplicity we consider the case of two or three independent variables. One particular solution of the stationary Schrödinger equation allows us to reduce this second-order equation to a linear first-order quaternionic differential equation. As in the one-dimensional case this is done with the aid of an auxiliary quaternionic Riccati equation. The resulting first-order quaternionic equation is equivalent to the static Maxwell system and is closely related to the Dirac equation. In the case of two independent variables it is the well-known Vekua equation from theory of pseudoanalytic (or generalized analytic) functions. Nevertheless, we show that even in this case it is very useful to consider not only complex valued functions, solutions of the Vekua equation, but complete quaternionic functions. In this way the first-order quaternionic equation represents two separate Vekua equations, one of which gives us solutions of the Schrödinger equation and the other one can be considered as an auxiliary equation of a simpler structure. Moreover for the auxiliary equation we always have the corresponding Bers generating pair (F, G), the base of the Bers theory of pseudoanalytic functions, and what is very important, the Bers derivatives of solutions of the auxiliary equation give us solutions of the main Vekua equation and as a consequence of the Schrödinger equation. Based on this fact we obtain an analogue of the Cauchy integral theorem for solutions of the stationary Schrödinger equation. Other results from theory of pseudoanalytic functions can be written for solutions of the Schrödinger equation. Moreover, for an ample class of potentials in the Schrödinger equation (which includes for instance all radial potentials), this new approach gives us a simple procedure allowing us to obtain an infinite sequence of solutions of the Schrödinger equation from one known particular solution.

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1. Introduction

Consider the one-dimensional static Schrödinger equation

$$u'' + vu = 0 \tag{1}$$

and the associated Riccati equation

$$y' + y^2 = -v. \tag{2}$$

Equation (1) is related to equation (2) by the easily inverted substitution

$$y = \frac{u'}{u}.$$

Thus solutions of the Riccati equation (2) are simply logarithmic derivatives of solutions of the Schrödinger equation (1) and vice versa solutions of (1) are logarithmic antiderivatives of solutions of (2). The generalization of this fact for a multidimensional situation was obtained in [8] (see also [10]). Among the peculiar properties of the Riccati equation stands out an important theorem of Euler, dating from 1760. If a particular solution y_0 of the Riccati equation is known, the substitution $y = y_0 + z$ reduces (2) to a Bernoulli equation which in turn is reduced by the substitution $z = \frac{1}{u}$ to a first-order linear equation. Thus given a particular solution of the Riccati equation, it can be linearized and the general solution can be found in two integrations. As a consequence of this, given a particular solution of the Schrödinger equation (1) the general solution can be found from a first-order linear equation. This can be seen immediately from the factorization of the one-dimensional Schrödinger operator

$$\partial^2 + v(x) = (\partial + y_0(x))(\partial - y_0(x)) \tag{3}$$

which is valid if and only if y_0 is a solution of (2).

In the present work we show that given a particular solution of a multidimensional stationary Schrödinger equation (we consider the case of three or two independent variables but this result can easily be generalized to n variables using Clifford algebras instead of quaternions) this equation of second order can be reduced to a first-order linear quaternionic differential equation. For doing this we use a quaternionic factorization of the Schrödinger operator proposed in [3, 4] (see also [7]) and the results on the quaternionic Riccati equation from [8, 10] where it was shown that having a particular solution of the quaternionic Riccati equation one can reduce it to a second-order linear equation. Here we show that the similarity with the one-dimensional situation is much closer, and one particular solution is sufficient to reduce the quaternionic Riccati equation to a first-order linear equation. The resulting first-order quaternionic equation is equivalent to the static Maxwell system and is closely related to the Dirac equation. In the case of two independent variables it is the well-known Vekua equation from theory of pseudoanalytic (or generalized analytic) functions (see, e.g., [2, 5, 6, 14, 15]). We show that even in this case it is very useful to consider not only complex valued functions, solutions of the Vekua equation, but complete quaternionic functions. In this way the first-order quaternionic equation represents two separate Vekua equations, one of which gives us solutions of the Schrödinger equation and the other one can be considered as an auxiliary equation of a simpler structure. Moreover for the auxiliary equation we always have in explicit form the corresponding Bers generating pair (F, G), the base of Bers' theory of pseudoanalytic functions and, which is very important, the Bers derivatives of solutions of the auxiliary equation give us solutions of the main Vekua equation and as a consequence of the Schrödinger equation. Based on this fact, for example, we obtain an analogue of the Cauchy integral theorem for solutions of the stationary Schrödinger equation. Other results from theory of pseudoanalytic functions can be written for solutions of the Schrödinger equation. Moreover, for an ample class of potentials in the Schrödinger equation (which includes for instance all radial potentials), this new approach gives us a simple procedure allowing us to obtain an infinite sequence of solutions of the Schrödinger equation from one known particular solution.

Besides this introduction the paper contains four sections. In section 2 we introduce necessary notation from quaternionic analysis. In section 3 we prove a spatial generalization of the Euler theorem for the Riccati equation and show how a particular solution of the Schrödinger equation allows us to reduce it to a first-order quaternionic equation. We observe that in the case of two independent variables this first-order equation represents two separate Vekua equations. In order to apply theory of pseudoanalytic functions to the resulting Vekua equations, in section 4 we introduce some necessary definitions and results from Bers' theory. Finally in section 5 we show how all the machinery of this quite forgotten mathematical theory allows us to obtain surprising results for the Schrödinger equation starting with an analogue of the Cauchy integral theorem and including infinite sequences of solutions generated by one particular solution.

2. Notation from quaternionic analysis

We will consider the algebra $\mathbb{H}(\mathbb{C})$ of complex quaternions or biquaternions which have the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where $\{q_k\} \subset \mathbb{C}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the quaternionic imaginary units.

The vectorial representation of a complex quaternion will be used. Namely, each complex quaternion q is a sum of a scalar q_0 and of a vector $\mathbf{q}$:

$$q = \mathrm{Sc}(q) + \mathrm{Vec}(q) = q_0 + \mathbf{q},$$

where $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. The purely vectorial complex quaternions ($\mathrm{Sc}(q) = 0$) are identified with vectors from $\mathbb{C}^3$. Note that $\mathbf{q}^2 = -\langle \mathbf{q}, \mathbf{q} \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product.

By M^p we denote the operator of multiplication by a complex quaternion p from the right-hand side: $M^p q = q \cdot p$.

More information on the structure of the algebra of complex quaternions can be found, for example, in [10] or [12].

Let q be a complex quaternion-valued differentiable function of $\mathbf{x} = (x_1, x_2, x_3)$. Denote

$$\mathrm{D}q = \mathbf{i}\frac{\partial}{\partial x_1}q + \mathbf{j}\frac{\partial}{\partial x_2}q + \mathbf{k}\frac{\partial}{\partial x_3}q.$$

This expression can be rewritten in a vector form as follows:

$$\mathrm{D}q = -\mathrm{div}\,\mathbf{q} + \mathrm{grad}\,q_0 + \mathrm{rot}\,\mathbf{q}.$$

That is, $\mathrm{Sc}(\mathrm{D}q) = -\mathrm{div}\,\mathbf{q}$ and $\mathrm{Vec}(\mathrm{D}q) = \mathrm{grad}\,q_0 + \mathrm{rot}\,\mathbf{q}$. Let us note that $\mathrm{D}^2 = -\Delta$.

If q_0 is a scalar function then $\mathrm{D}q_0$ coincides with $\mathrm{grad}\,q_0$. The expression $\mathrm{D}q_0/q_0$ will be called the logarithmic derivative of q_0.

3. Reduction of the Schrödinger equation to a first-order quaternionic equation

Consider the equation

$$(-\Delta + u) f = 0 \tag{4}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, f and u are complex valued functions. We assume that f is twice differentiable. Together with (4) we introduce the following quaternionic equation,

$$D\mathbf{q} + \mathbf{q}^2 = -u \tag{5}$$

where $\mathbf{q}$ is a purely vectorial differentiable biquaternion valued function.

Theorem 1 [3]**.** *For an arbitrary scalar twice differentiable function f the following equality holds,*

$$(\mathrm{D} + M^{\mathbf{h}})(\mathrm{D} - M^{\mathbf{h}}) f = (-\Delta + u) f \tag{6}$$

if and only if $\mathbf{h}$ is a solution of (5).

Thus, given a particular solution of (5) the Schrödinger operator in (4) can be factorized.

Theorem 2 [8]**.** *Solutions of (4) are related to solutions of (5) in the following way. For any nonvanishing solution f of (4) its logarithmic derivative*

$$\mathbf{q} = \frac{\mathrm{D}f}{f} \tag{7}$$

is a solution of (5) and any solution $\mathbf{q}$ of (5) is a logarithmic derivative of form (7) of a solution of (4).

Proof. A direct substitution into equation (5) shows us that for a nonvanishing solution f of (4) its logarithmic derivative (7) is a solution of (5). Now let us suppose that $\mathbf{q}$ is a solution of (5). From the vector part of (5) we have that $\mathbf{q}$ is a gradient of some scalar function ξ: $\mathbf{q} = \operatorname{grad} \xi$. Then $\mathbf{q}$ can be represented in the form (7) where $f = \mathrm{e}^{\xi}$. Substituting (7) into (5) we obtain that f is a solution of (4). □

Remark 3. Theorems 1 and 2 show us that equation (5) is a generalization of the Riccati equation. We will call it quaternionic Riccati equation.

Lemma 4 [9]**.** *For a nonvanishing scalar differentiable function ε there exists a one-to-one correspondence between solutions of the static Maxwell system*

$$\operatorname{div}(\varepsilon\,\mathbf{E}) = 0, \tag{8}$$

$$\operatorname{rot}\mathbf{E} = 0, \tag{9}$$

and solutions of the equation

$$(\mathrm{D} + M^{\mathbf{h}})\mathbf{F} = 0 \tag{10}$$

where

$$\mathbf{h} = \frac{\mathrm{D}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}.$$

Vector $\mathbf{E}$ is a solution of (8), (9) if and only if the vector $\mathbf{F} = \sqrt{\varepsilon}\mathbf{E}$ is a solution of (10).

Proof. System (8), (9) can be rewritten in the form

$$\mathrm{D}\mathbf{E} = \left\langle \frac{\operatorname{grad}\varepsilon}{\varepsilon}, \mathbf{E} \right\rangle.$$

Let us make a simple observation: the scalar product of two vectors $\mathbf{p}$ and $\mathbf{q}$ can be written as follows:

$$\langle \mathbf{p}, \mathbf{q} \rangle = -\tfrac{1}{2}({}^{\mathbf{p}}M + M^{\mathbf{p}})\mathbf{q}.$$

Then we have

$$\left(\mathrm{D} + \frac{1}{2}\frac{\operatorname{grad}\varepsilon}{\varepsilon}\right)\mathbf{E} = -\frac{1}{2}M^{\frac{\operatorname{grad}\varepsilon}{\varepsilon}}\mathbf{E}. \tag{11}$$

Note that

$$\frac{1}{2}\frac{\operatorname{grad}\varepsilon}{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}.$$

Then equation (11) can be rewritten in the following form,

$$\frac{1}{\sqrt{\varepsilon}}\mathrm{D}(\sqrt{\varepsilon}\mathbf{E}) + \mathbf{Eh} = 0 \tag{12}$$

where $\mathbf{h} = \mathrm{D}\sqrt{\varepsilon}/\sqrt{\varepsilon}$. Introducing the notation $\mathbf{F} = \sqrt{\varepsilon}\mathbf{E}$ and multiplying (12) by $\sqrt{\varepsilon}$ we obtain the equivalence of the system (8), (9) to (10). □

Lemma 5 [8]. *Let* $\mathbf{h}$ *be an arbitrary particular solution of (5) (then as was mentioned above it is a gradient of some scalar function* ξ*). The general solution of (5) has the form*

$$\mathbf{q} = \mathbf{h} + \mathbf{g}, \tag{13}$$

where $\mathbf{g} = (\operatorname{grad}\Psi)/\Psi$ *and* Ψ *is a general solution of the equation*

$$\triangle\Psi + 2\langle \operatorname{grad}\xi, \operatorname{grad}\Psi \rangle = 0, \tag{14}$$

or equivalently of

$$\operatorname{div}(\mathrm{e}^{2\xi}\operatorname{grad}\Psi) = 0. \tag{15}$$

Proof. Substituting (13) into (5) gives

$$\mathrm{D}\mathbf{g} - 2\langle \mathbf{h}, \mathbf{g} \rangle + \mathbf{g}^2 = 0. \tag{16}$$

Note that the vector part of (16) is $\operatorname{rot}\mathbf{g} = 0$, so that

$$\mathbf{g} = \operatorname{grad}\Phi$$

for some function Φ. If $\Psi = \mathrm{e}^{\Phi}$, this is equivalent to

$$\mathbf{g} = (\operatorname{grad}\Psi)/\Psi.$$

Equation (16), written in terms of Ψ, is

$$-\frac{1}{\Psi^2}(\operatorname{grad}\Psi)^2 - \frac{1}{\Psi}\triangle\Psi - \frac{2}{\Psi}\langle \operatorname{grad}\xi, \operatorname{grad}\Psi \rangle + \frac{1}{\Psi^2}(\operatorname{grad}\Psi)^2 = 0,$$

so that (16) is equivalent to

$$\triangle\Psi + 2\langle \operatorname{grad}\xi, \operatorname{grad}\Psi \rangle = 0.$$

Noting that

$$\operatorname{div}(\mathrm{e}^{2\xi}\operatorname{grad}\Psi) = 2\,\mathrm{e}^{2\xi}\langle \operatorname{grad}\xi, \operatorname{grad}\Psi \rangle + \mathrm{e}^{2\xi}\triangle\Psi = \mathrm{e}^{2\xi}(\triangle\Psi + 2\langle \operatorname{grad}\xi, \operatorname{grad}\Psi \rangle),$$

this equation can be rewritten in the form

$$\operatorname{div}(\mathrm{e}^{2\xi}\operatorname{grad}\Psi) = 0.$$ □

Now we are ready to prove a generalization of the Euler theorem for the quaternionic Riccati equation.

Theorem 6 (Euler's theorem for the quaternionic Riccati equation)**.** *Let* $\mathbf{h} = \operatorname{grad} \xi$ *be a particular solution of (5). The general solution of the quaternionic Riccati equation has the form* $\mathbf{q} = \mathbf{h} + \mathbf{g}$ *where* $\mathbf{g} = \frac{D\Psi}{\Psi}$ *and* Ψ *is obtained from the equation*

$$\operatorname{grad} \Psi = \mathrm{e}^{-\xi} \mathbf{F} \tag{17}$$

where $\mathbf{F}$ *is the general solution of (10).*

Proof. According to lemma 5 it is sufficient to prove that Ψ is a solution of (14) (or what is the same of (15)) if and only if the vector $\mathbf{F} = \mathrm{e}^{\xi} \operatorname{grad} \Psi$ is a solution of (10). Let us note that if Ψ is a solution of (14) then the vector $\mathbf{E} = \operatorname{grad} \Psi$ is a solution of the system (8), (9) where $\varepsilon = \mathrm{e}^{2\xi}$ and vice versa if $\mathbf{E}$ is a solution of (8), (9) then it is a gradient of some function Ψ which is necessarily a solution of (14). Now due to lemma 4, $\operatorname{grad} \Psi$ is a solution of (8), (9) if and only if $\mathbf{F} = \mathrm{e}^{\xi} \operatorname{grad} \Psi$ is a solution of equation (10) where $\mathbf{h} = \frac{D\mathrm{e}^{\xi}}{\mathrm{e}^{\xi}} = \operatorname{grad} \xi$. □

Thus, given a particular solution of the quaternionic Riccati equation, the general solution reduces to the linear first-order equation (10), exactly as in the one-dimensional situation.

Using theorem 2 we immediately arrive at the following result for the Schrödinger equation.

Theorem 7. *Let* f_0 *be a nonvanishing particular solution of (4) and* $\mathbf{F}$ *be the general solution of (10) where* $\mathbf{h} = \mathrm{D} f_0 / f_0$*. Then the general solution* f *of (4) is the logarithmic antiderivative of* $\mathbf{q}$*:* $\frac{\mathrm{D} f}{f} = \mathbf{q}$*, where* $\mathbf{q} = \mathbf{h} + \mathbf{g}$, $\mathbf{g} = \frac{\mathrm{D}\Psi}{\Psi}$ *and* Ψ *is obtained from the equation*

$$\operatorname{grad} \Psi = \frac{\mathbf{F}}{f_0}. \tag{18}$$

Proof. From theorem 2 we have that f is the logarithmic antiderivative of $\mathbf{q}$ (equation (7)), where $\mathbf{q}$ is the general solution of (5). For a particular solution f_0 of (4) the vector

$$\mathbf{h} = \frac{\mathrm{D} f_0}{f_0} = \operatorname{grad} \ln f_0$$

is a particular solution of (5). Due to theorem 6 the general solution of (5) has the form $\mathbf{q} = \mathbf{h} + \mathbf{g}$, where $\mathbf{g} = \frac{\mathrm{D}\Psi}{\Psi}$ and Ψ is obtained from the equation

$$\operatorname{grad} \Psi = \mathrm{e}^{-\ln f_0} \mathbf{F} = \frac{\mathbf{F}}{f_0}$$

where $\mathbf{F}$ is the general solution of (10). □

Remark 8. The logarithmic antiderivative of $\mathbf{q}$ always exists and can be obtained easily. Being a solution of (5) $\mathbf{q}$ is necessarily a gradient of some function Φ which can be constructed analytically. Then f has the form $f = C\mathrm{e}^{\Phi}$, where C is a complex constant.

Remark 9. From theorem 7 it follows that for any vector $\mathbf{F}$, solution of (10) with $\mathbf{h} = \mathrm{D} f_0 / f_0$, the vector $\mathbf{F}/f_0$ must be a gradient of some scalar function Ψ, and this is true. Let us show that indeed $\operatorname{rot}(\mathbf{F}/f_0) = 0$. Note that this condition is equivalent to the equality $\operatorname{Vec}(\mathrm{D}(\mathbf{F}/f_0)) = 0$. Consider

$$\mathrm{D}\left(\frac{\mathbf{F}}{f_0}\right) = \frac{1}{f_0}\mathrm{D}\mathbf{F} - \frac{\mathrm{D} f_0}{f_0^2}\mathbf{F} = -\left(\mathbf{F}\frac{\mathrm{D} f_0}{f_0^2} + \frac{\mathrm{D} f_0}{f_0^2}\mathbf{F}\right) = 2\left\langle \mathbf{F}, \frac{\mathrm{D} f_0}{f_0^2}\right\rangle.$$

Thus $\operatorname{Vec}(\mathrm{D}(\mathbf{F}/f_0)) = 0$ and hence the vector $\mathbf{F}/f_0$ is a gradient.

Remark 10. Let us summarize the results of this section in the following chain of actions which one should follow in order to obtain solutions of (4) from solutions of (10).

Given a nonvanishing particular solution f_0 of the Schrödinger equation (4) we construct the vector $\mathbf{h} = \mathrm{D} f_0/f_0$ and consider equation (10). Taking a solution $\mathbf{F}$ of (10) we find Ψ from (18). Then we construct the vectors $\mathbf{g} = \mathrm{D}\Psi/\Psi$ and $\mathbf{q} = \mathbf{h} + \mathbf{g}$. $\mathbf{q}$ is necessarily a gradient of some scalar function Φ. Finding Φ we finally obtain a solution of (4) as $f = C\mathrm{e}^{\Phi}$, where C is a complex constant.

The same chain of actions can be expressed as a single formula in the following way. Let $\mathbf{G}$ be a complex valued vector such that $\operatorname{rot}\mathbf{G} \equiv 0$. Then the complex valued scalar function φ is said to be its antigradient if $\operatorname{grad}\varphi = \mathbf{G}$. We will write $\varphi = \mathcal{A}[\mathbf{G}]$. The operator $\mathcal{A}$ is a simple generalization of the usual antiderivative and it defines the function φ up to an arbitrary constant. Its explicit representation is well known and has the form

$$\mathcal{A}[\mathbf{G}](x, y, z) = \int_{x_0}^{x} G_1(\xi, y_0, z_0)\,\mathrm{d}\xi + \int_{y_0}^{y} G_2(x, \zeta, z_0)\,\mathrm{d}\zeta + \int_{z_0}^{z} G_3(x, y, \eta)\,\mathrm{d}\eta + C.$$

Then according to our chain of actions we have

$$\Psi = \mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right], \qquad \mathbf{g} = \left(\mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right]\right)^{-1}\cdot\frac{\mathbf{F}}{f_0}, \qquad \mathbf{q} = \frac{\mathrm{D}f_0}{f_0} + \left(\mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right]\right)^{-1}\cdot\frac{\mathbf{F}}{f_0};$$

$$\Phi = \mathcal{A}\left[\frac{\mathrm{D}f_0}{f_0} + \left(\mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right]\right)^{-1}\cdot\frac{\mathbf{F}}{f_0}\right]$$

and finally

$$f = \exp\left(\mathcal{A}\left[\frac{\mathrm{D}f_0}{f_0} + \left(\mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right]\right)^{-1}\cdot\frac{\mathbf{F}}{f_0}\right]\right).$$

As $\mathrm{e}^{\mathcal{A}[\mathrm{D}f_0/f_0]} = C_1 f_0$ we obtain

$$f = C_1 f_0 \exp\left(\mathcal{A}\left[\left(\mathcal{A}\left[\frac{\mathbf{F}}{f_0}\right]\right)^{-1}\cdot\frac{\mathbf{F}}{f_0}\right]\right). \tag{19}$$

Thus given a particular solution of the Schrödinger equation (4), due to (19) the general solution reduces to the first-order equation (10). It is interesting to note that due to lemma 4, equation (10) is equivalent to the static Maxwell system. It is closely related (see [7, 11]) also to the Dirac equation as well as to the Beltrami fields which are solutions of the equation $\operatorname{rot}\mathbf{f} + \alpha\mathbf{f} = 0$ (see, e.g., [1, 13]).

The following statement gives us the way to transform solutions of the Schrödinger equation into solutions of (10).

Proposition 11. *Let f_1 be another nonvanishing solution of the Schrödinger equation (4). Then the ratio $\Psi = f_1/f_0$ is a solution of the equation*

$$\operatorname{div}\left(f_0^2 \operatorname{grad}\Psi\right) = 0 \tag{20}$$

and the vector $\mathbf{F} = f_0\mathrm{D}(f_1/f_0)$ is a solution of equation (10) where $\mathbf{h} = \mathrm{D}f_0/f_0$.

Proof. By theorem 2 the vector $\mathbf{q} = \mathrm{D}f_1/f_1$ is a solution of (5). Then consider the vector $\mathbf{g}$ from lemma 5:

$$\mathbf{g} = \mathbf{q} - \mathbf{h} = \frac{\mathrm{D}f_1}{f_1} - \frac{\mathrm{D}f_0}{f_0} = \mathrm{D}(f_1/f_0)/(f_1/f_0).$$

Thus we have that $\mathbf{g} = \mathrm{D}\Psi/\Psi$ where $\Psi = f_1/f_0$, and by lemma 5, Ψ satisfies (20).

From (18) we obtain that $\mathbf{F} = f_0\mathrm{D}\Psi = f_0\mathrm{D}(f_1/f_0)$ is a solution of (10). □

Let us consider equation (10) for a biquaternion valued function p whose scalar part is not necessarily zero

$$(\mathrm{D} + M^{\mathbf{h}})p = 0 \tag{21}$$

and let us use the following representation for biquaternions:

$$p = P_1 + P_2\mathbf{j},$$

where $P_1 = p_0 + p_3\mathbf{k}$ and $P_2 = p_2 - p_1\mathbf{k}$. Then $\mathrm{D} = D_1 + D_2\mathbf{j}$, where $D_1 = \partial_3\mathbf{k}$, $D_2 = \partial_2 - \partial_1\mathbf{k}$ and $\mathbf{h} = H_1 + H_2\mathbf{j}$, where $H_1 = h_3\mathbf{k}$, $H_2 = h_2 - h_1\mathbf{k}$. Using these notations, equation (21) can be rewritten as the following system:

$$D_1P_1 - D_2\overline{P}_2 + H_1P_1 - \overline{H}_2P_2 = 0, \tag{22}$$

$$D_2\overline{P}_1 + D_1P_2 + H_2P_1 + \overline{H}_1P_2 = 0. \tag{23}$$

Now let us suppose that both p and $\mathbf{h}$ do not depend on x_3. Then system (22), (23) turns into the pair of decoupled equations:

$$\overline{D}_2P_1 + \overline{H_2P_1} = 0 \tag{24}$$

and

$$\overline{D}_2P_2 + H_2\overline{P}_2 = 0 \tag{25}$$

which are nothing but Vekua's equations describing pseudoanalytic or generalized analytic functions (see, e.g., [5, 15]).

Remark 12. Here we should mention that in general the components $p_0, \ldots, p_3$ as well as h_1, h_2 can be complex valued functions, hence P_1, P_2 and H_2 can be bicomplex. Nevertheless this detail is insignificant for what follows, because all results from Bers' theory which will be used in the subsequent sections are valid for bicomplex solutions also. Of course, when u in (4) is a real valued function we can consider real valued solutions of (4) only. In that case (24) and (25) are usual Vekua's equations.

Remark 13. In what follows we consider u and f in (4) being independent of x_3. Then given a particular solution f_0 of (4), the general solution reduces to equation (25) (which in this case is equivalent to (10)). Thus we are primarily interested in solutions of (25), and (24) can be considered as an auxiliary equation. Nevertheless as we will see in section 5 equations (24) and (25) are closely related to each other. With the aid of Bers' theory solutions of (25) can be obtained from solutions of (24) and it is interesting to note that by construction we always have at least two solutions of (24) in explicit form. It is easy to see that the functions

$$F = \frac{1}{f_0} \qquad \text{and} \qquad G = f_0\mathbf{k}$$

are solutions of (21) where $\mathbf{h} = \mathrm{D}f_0/f_0$ and consequently they are solutions of (24).

4. Some definitions and results from Bers' theory

Bers' theory of pseudoanalytic functions was essentially developed in [5] (see also [6]). It is based on the so-called generating pair, a pair of complex functions F and G satisfying the inequality

$$\mathrm{Im}(\overline{F}G) > 0 \tag{26}$$

in some domain of interest Ω which may coincide with the whole complex plane. F and G are assumed to possess partial derivatives with respect to the real variables x and y. In this case the operators $\partial_{\overline{z}} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}$ can be applied (usually these operators are introduced with the factor $1/2$, nevertheless here it is somewhat more convenient to consider them without it) and the following characteristic coefficients of the pair (F, G) can be defined,

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G},$$

where the subindex $\overline{z}$ or z means the application of $\partial_{\overline{z}}$ or ∂_z respectively.

Every complex function w defined in a subdomain of Ω admits the unique representation $w = \phi F + \psi G$ where the functions ϕ and ψ are real valued. Sometimes it is convenient to associate with the function w the function $\omega = \phi + \mathrm{i}\psi$. The correspondence between w and ω is one-to-one.

Bers introduces the notion of the (F, G) derivative of a function w which exists and has the form

$$\dot{w} = \phi_z F + \psi_z G = w_z - A_{(F,G)}w - B_{(F,G)}\overline{w} \tag{27}$$

if and only if

$$\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \tag{28}$$

This last equation can be rewritten in the following form,

$$w_{\overline{z}} = a_{(F,G)}w + b_{(F,G)}\overline{w}$$

which we call the Vekua equation. Solutions of this equation are called (F, G)-pseudoanalytic functions. If w is (F, G)-pseudoanalytic, the associated function ω is called (F, G)-pseudoanalytic of second kind.

Remark 14. The functions F and G are (F, G)-pseudoanalytic, and $\dot{F} \equiv \dot{G} \equiv 0$.

Definition 15. *Let (F, G) and (F_1, G_1) be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \text{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 16. *Let w be an (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot{w}$ is an (F_1, G_1)-pseudoanalytic function.*

Definition 17. *Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is defined by the formulae*

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

Theorem 18.

$$(F, G)^{**} = (F, G),$$

$$a_{(F^*,G^*)} = -a_{(F,G)}, \qquad A_{(F^*,G^*)} = -A_{(F,G)},$$

$$b_{(F^*,G^*)} = -\overline{B}_{(F,G)}, \qquad B_{(F^*,G^*)} = -\overline{b}_{(F,G)}.$$

Lemma 19. *If* (F_1, G_1) *is a successor of* (F, G) *then* $(F, G)^*$ *is a successor of* $(F_1, G_1)^*$.

The (F, G) integral of w on a rectifiable curve Γ is, by definition,

$$\int_\Gamma w \, \mathrm{d}_{(F,G)} z = \mathrm{Re} \int_\Gamma F^* w \, \mathrm{d}z - \mathrm{i} \, \mathrm{Re} \int_\Gamma G^* w \, \mathrm{d}z.$$

Another important integral is also needed

$$* \int_\Gamma w \, \mathrm{d}_{(F,G)} z = \mathrm{Re} \int_\Gamma G^* w \, \mathrm{d}z + \mathrm{i} \, \mathrm{Re} \int_\Gamma F^* w \, \mathrm{d}z$$

(we follow the notation of Bers).

A continuous function w defined in a domain Ω is called (F, G)-integrable if for every closed curve Γ situated in a simply connected subdomain of Ω,

$$\int_\Gamma w \, \mathrm{d}_{(F,G)} z = 0.$$

Theorem 20. *An* (F, G) *derivative* $\dot{w}$ *of an* (F, G)*-pseudoanalytic function* w *is* (F, G)*-integrable and* $* \int_{z_0}^{z_1} \dot{w} \, \mathrm{d}_{(F,G)} z = \omega(z_1) - \omega(z_0)$.

The integral $* \int_{z_0}^{z_1} \dot{w} \, \mathrm{d}_{(F,G)} z$ is called (F, G)-antiderivative of $\dot{w}$.

Theorem 21. *Let* (F, G) *be a predecessor of* (F_1, G_1). *A continuous function is* (F_1, G_1)*-pseudoanalytic if and only if it is* (F, G)*-integrable.*

5. Applications of Bers' theory to the stationary Schrödinger equation

Let us return to equations (24) and (25) which in a two-dimensional case are equivalent to the quaternionic equation (21). In order to use Bers' notation from the preceding section we rewrite (24) and (25) in the following form,

$$w_{\overline{z}} = b\overline{w} \tag{29}$$

and

$$v_{\overline{z}} = \overline{b}\overline{v} \tag{30}$$

where $z = x + \mathrm{i}y$, $x = x_2$, $y = x_1$ and instead of the imaginary unit $\mathbf{k}$ we write i. It is easy to see that $b = -\overline{H}_2 = -\partial_{\overline{z}} f_0 / f_0$ and $w = P_1$, $v = P_2$.

As was mentioned above (remark 13) for equation (29) we know always two solutions

$$F = \frac{1}{f_0} \qquad \text{and} \qquad G = \mathrm{i} f_0, \tag{31}$$

which obviously fulfil (26). Thus (F, G) is a generating pair corresponding to equation (29). We have

$$a_{(F,G)} = 0, \qquad b_{(F,G)} = b = -\frac{\partial_{\overline{z}} f_0}{f_0},$$

$$A_{(F,G)} = 0, \qquad B_{(F,G)} = -\frac{\partial_z f_0}{f_0}.$$

According to definition 15 the characteristic coefficients for a successor of (F, G) have the form

$$a_{(F_1,G_1)} = 0, \qquad b_{(F_1,G_1)} = \frac{\partial_z f_0}{f_0} = -\overline{b}.$$

Then due to theorem 16, if w is a solution of (29) then its (F, G) derivative is a solution of the equation

$$W_{\overline{z}} = -\overline{bW}, \tag{32}$$

but solutions of the last equation multiplied by i become solutions of (30) and vice versa. Thus we obtain the following statement.

Theorem 22. *Let w be a solution of (29). Then the function*

$$v = \mathrm{i}\dot{w} = \mathrm{i}\left(w_z + \frac{\partial_z f_0}{f_0}\overline{w}\right)$$

is a solution of (30).

It is easy to see that according to definition 17:

$$F^* = -\frac{\mathrm{i}}{f_0}, \qquad G^* = f_0$$

and

$$b_{(F^*,G^*)} = -\overline{B}_{(F,G)} = -b.$$

Thus the (F, G) integral of a function W is defined as follows:

$$\begin{aligned}\int_\Gamma W\,\mathrm{d}_{(F,G)}z &= -\mathrm{Re}\int_\Gamma \frac{\mathrm{i}}{f_0} W\,\mathrm{d}z - \mathrm{i}\,\mathrm{Re}\int_\Gamma f_0 W\,\mathrm{d}z \\ &= \mathrm{Im}\int_\Gamma \frac{W}{f_0}\,\mathrm{d}z - \mathrm{i}\,\mathrm{Re}\int_\Gamma f_0 W\,\mathrm{d}z.\end{aligned}$$

From theorems 20 and 21 we obtain the following result.

Theorem 23. *Let v be a solution of (30) in a domain Ω. Then for every closed curve Γ situated in a simply connected subdomain of Ω,*

$$\mathrm{Re}\int_\Gamma \frac{v}{f_0}\,\mathrm{d}z + \mathrm{i}\,\mathrm{Im}\int_\Gamma f_0 v\,\mathrm{d}z = 0. \tag{33}$$

Proof. For any solution v of (30) the function $W = \mathrm{i}v$ is a solution of (32). As (32) corresponds to a successor of (F, G), by theorem 20 W is (F, G)-integrable. That is

$$\mathrm{Im}\int_\Gamma \frac{W}{f_0}\,\mathrm{d}z - \mathrm{i}\,\mathrm{Re}\int_\Gamma f_0 W\,\mathrm{d}z = 0.$$

Now substituting $\mathrm{i}v$ instead of W we obtain (33). □

In order to analyse the meaning of this result for solutions of the Schrödinger equation let us rewrite some statements from section 3 in our 'two-dimensional' notation.

Consider the equation

$$(-\partial_z\partial_{\overline{z}} + u)f = 0 \tag{34}$$

where u and f depend on x, y, and $z = x + \mathrm{i}y$. For simplicity we consider u and f being real-valued functions. The corresponding Riccati equation (5) takes the form

$$\partial_{\overline{z}} Q + |Q|^2 = u$$

where $\mathbf{q} = Q\mathbf{j}$. Equation (10) turns into (30), $\mathbf{F}$ from (10) and v from (30) are related by the equality $\mathbf{F} = v\mathbf{j}$. Then theorem 7 in a two-dimensional situation can be rewritten as follows.

Theorem 24. *Let f_0 be a nonvanishing particular solution of (34) and v be the general solution of (30) where $b = -\partial_{\bar{z}} f_0/f_0$. Then the general solution f of (34) is obtained from the equation $\partial_z f = Qf$, where $Q = \partial_z f_0/f_0 + \partial_z \Psi/\Psi$ and Ψ is obtained from the equation $\partial_z \Psi = v/f_0$.*

As was explained in remarks 8–10, given a solution v of (30), the corresponding solution f of (34) can be constructed analytically. The procedure consists of various simple steps, on two of which it requires reconstruction of the potential function from its gradient.

From proposition 11 we obtain the following statement.

Proposition 25. *Let f_1 be another solution of (34). Then the function*

$$v = f_0 \partial_z (f_1/f_0) \tag{35}$$

is a solution of (30), where $b = -\partial_{\bar{z}} f_0/f_0$.

Having this precise relation between solutions of (34) and (30) we are able to prove the following result.

Theorem 26 (Cauchy's integral theorem for the Schrödinger equation)**.** *Let f_0 be a nonvanishing solution of (34) in a domain Ω and f_1 be another arbitrary solution of (34) in Ω. Then for every closed curve Γ situated in a simply connected subdomain of Ω,*

$$\operatorname{Re} \int_\Gamma \partial_z \left(\frac{f_1}{f_0} \right) \mathrm{d}z + \mathrm{i} \operatorname{Im} \int_\Gamma f_0^2 \partial_z \left(\frac{f_1}{f_0} \right) \mathrm{d}z = 0. \tag{36}$$

Proof. Substitution of (35) into (33) gives us the result. □

Remark 27. This theorem is also valid when $u \equiv 0$, that is for f_0 and f_1 being harmonic functions. If we take $f_0 \equiv 1$, then (36) turns into the equality $\int_\Gamma \partial_z f_1 \, \mathrm{d}z = 0$ which is obviously true because if f_1 is harmonic, then $\partial_z f_1$ is analytic. In example 32 we will give a nontrivial example illustrating this theorem.

From theorems 21 and 26 we obtain an analogue of the Morera theorem for the Schrödinger equation (34).

Theorem 28. *Let f_0 be a nonvanishing particular solution of (34). The function f_1 is a solution of (34) also if (36) is valid for every closed curve Γ situated in a simply connected subdomain of Ω.*

Consider equation (28) with the functions (31). It takes the form

$$\phi_{\bar{z}} + \mathrm{i} f_0^2 \psi_{\bar{z}} = 0. \tag{37}$$

Proposition 29. *Let the function $w = \phi F + \psi G$ be (F, G)-pseudoanalytic corresponding to the functions (31), that is*

$$w = \frac{\phi}{f_0} + \mathrm{i} f_0 \psi \tag{38}$$

is a solution of (29). Then

$$\langle \operatorname{grad} \phi, \operatorname{grad} \psi \rangle = 0.$$

Proof. Function (38) is a solution of (29) if and only if (37) is valid. Let us rewrite (37) in the form

$$\phi_x - f_0^2 \psi_y = 0 \tag{39}$$

$$\phi_y + f_0^2 \psi_x = 0. \tag{40}$$

From (40) we have $f_0^2 = -\phi_y/\psi_x$. Substituting this expression into (39) we obtain

$$\phi_x + \psi_y \frac{\phi_y}{\psi_x} = 0$$

or

$$\phi_x \psi_x + \phi_y \psi_y = 0.$$

□

In general, solution of (37) or equivalently of system (39), (40) seems to be a difficult task. Nevertheless for a quite general class of functions f_0 we can obtain solutions of (39), (40) explicitly. Let us make the following suppositions.

Condition 30. Let f_0 be a function of some variable $\rho : f_0 = f_0(\rho)$ such that $\frac{\Delta\rho}{|\mathrm{grad}\,\rho|^2}$ is a function of ρ. We denote it by $s(\rho) = \frac{\Delta\rho}{|\mathrm{grad}\,\rho|^2}$.

The simplest example of such ρ is of course any harmonic function. Another important example is $\rho = \sqrt{x^2 + y^2}$.

Consider system (39), (40) and look for ϕ being a function of $\rho : \phi = \phi(\rho)$ (as we show below such solution always exists). Then

$$\psi_x = -\frac{\rho_y}{f_0^2}\phi', \qquad \psi_y = \frac{\rho_x}{f_0^2}\phi'. \tag{41}$$

For the solubility of this system we obtain the following condition,

$$\frac{\partial}{\partial x}\left(\frac{\rho_x}{f_0^2}\phi'\right) + \frac{\partial}{\partial y}\left(\frac{\rho_y}{f_0^2}\phi'\right) = 0$$

which can be written as an ordinary differential equation

$$\phi'' + \left(s - 2\frac{f_0'}{f_0}\right)\phi' = 0.$$

From here we have

$$\phi'(\rho) = \mathrm{e}^{-S(\rho)} f_0^2(\rho)$$

where $S(\rho) = \int s(\rho)\,\mathrm{d}\rho$.

With the aid of (41) we can reconstruct ψ. Nevertheless we are interested neither in ψ nor in ϕ but in ϕ_z and ψ_z instead. Having them we construct the function $v = \mathrm{i}(\phi_z F + \psi_z G) = \mathrm{i}(\phi_z/f_0 + \mathrm{i}\psi_z f_0)$ which gives us a solution of (30). We have

$$\phi_z = \phi'\rho_z = \mathrm{e}^{-S} f_0^2 \rho_z \tag{42}$$

and

$$\psi_z = -\phi'\left(\frac{\rho_y + \mathrm{i}\rho_x}{f_0^2}\right) = -\mathrm{i}\mathrm{e}^{-S}\rho_z. \tag{43}$$

Then we obtain the following solution of (30):

$$v_1 = \mathrm{i}(\phi_z F + \psi_z G) = 2\,\mathrm{i} f_0\,\mathrm{e}^{-S}\rho_z.$$

In much the same way we can construct another solution of (30) looking for $\psi = \psi(\rho)$. Then

$$\phi_x = f_0^2 \rho_y \psi', \qquad \phi_y = -f_0^2 \rho_x \psi'.$$

and $\psi' = \mathrm{e}^{-S}\big/ f_0^2$. Calculating ϕ_z and ψ_z we obtain

$$\phi_z = \mathrm{i}\mathrm{e}^{-S}\rho_z \tag{44}$$

and

$$\psi_z = \frac{\mathrm{e}^{-S}}{f_0^2}\rho_z. \tag{45}$$

Thus we arrive at the following solution of (30):

$$v_2 = \mathrm{i}\left(\frac{\mathrm{i}\mathrm{e}^{-S}\rho_z}{f_0} + \mathrm{i}\frac{\mathrm{e}^{-S}}{f_0^2}\rho_z f_0\right) = -2\,\mathrm{e}^{-S}\frac{\rho_z}{f_0}.$$

Denote

$$F_I = \frac{v_1}{2} \qquad \text{and} \qquad G_I = \frac{v_2}{2}. \tag{46}$$

Then $\mathrm{Im}(\overline{F}_I G_I) = \mathrm{e}^{-2S}\,|\mathrm{grad}\,\rho|^2 > 0$ and hence we have a generating pair for (30) in explicit form. Note that (F_I, G_I) is not a successor of (F, G) but a successor multiplied by i: $(F_I, G_I) = \mathrm{i}(F_1, G_1)$.

It is interesting to see what are the new solutions f_1 and f_2 of the Schrödinger equation (34) corresponding to F_I and G_I. In order to illustrate this let us consider an example.

Example 31. Let $u(x, y) = x^2 + y^2$. Then a particular solution of (34) can be chosen in the form

$$f_0(x, y) = \mathrm{e}^{xy}.$$

Obviously $\rho(x, y) = xy$ being a harmonic function satisfies condition 30. Then $F_I = \mathrm{i}\mathrm{e}^{xy}(y - \mathrm{i}x)$ and $G_I = -\mathrm{e}^{-xy}(y - \mathrm{i}x)$. Returning to the notation of section 3 (see the beginning of section 5) we have $F_I = \mathbf{k}\,\mathrm{e}^{x_1x_2}(x_1 - \mathbf{k}x_2) = \mathrm{e}^{x_1x_2}(x_2 + \mathbf{k}x_1)$ and $G_I = -\mathrm{e}^{-x_1x_2}(x_1 - \mathbf{k}x_2)$. Then the corresponding pair of solutions of (10) has the form

$$\mathbf{F}_1 = F_I\mathbf{j} = \mathrm{e}^{x_1x_2}(-x_1\mathbf{i} + x_2\mathbf{j}) \qquad \text{and} \qquad \mathbf{F}_2 = G_I\mathbf{j} = -\mathrm{e}^{-x_1x_2}(x_2\mathbf{i} + x_1\mathbf{j}).$$

Next step (see remark 10) consists in finding the corresponding functions Ψ_1 and Ψ_2 from equation (18). Thus we should reconstruct Ψ_1 and Ψ_2 from the equalities

$$\mathrm{grad}\,\Psi_1 = -x_1\mathbf{i} + x_2\mathbf{j} \qquad \text{and} \qquad \mathrm{grad}\,\Psi_2 = -\mathrm{e}^{-2x_1x_2}(x_2\mathbf{i} + x_1\mathbf{j}).$$

Using the standard formula for finding the potential function from its gradient we obtain

$$\Psi_1 = -\tfrac{1}{2}\left(x_1^2 - x_2^2 - C_1\right) \qquad \text{and} \qquad \Psi_2 = \tfrac{1}{2}(\mathrm{e}^{-2x_1x_2} + C_2)$$

where C_1 and C_2 are arbitrary constants.

Now we can construct the vectors $\mathbf{g}_1 = \mathrm{grad}\,\Psi_1/\Psi_1$ and $\mathbf{g}_2 = \mathrm{grad}\,\Psi_2/\Psi_2$:

$$\mathbf{g}_1 = \frac{2}{x_1^2 - x_2^2 - C_1}(x_1\mathbf{i} - x_2\mathbf{j}), \qquad \mathbf{g}_2 = 2\left(\frac{C_2}{\mathrm{e}^{-2x_1x_2} + C_2} - 1\right)(x_2\mathbf{i} + x_1\mathbf{j}).$$

Noting that $\mathbf{h} = Df_0/f_0 = x_2\mathbf{i} + x_1\mathbf{j}$ we obtain two solutions $\mathbf{q}_1 = \mathbf{h} + \mathbf{g}_1$ and $\mathbf{q}_2 = \mathbf{h} + \mathbf{g}_2$ for (5):

$$\mathbf{q}_1 = \left(x_2 + \frac{2x_1}{x_1^2 - x_2^2 - C_1}\right)\mathbf{i} + \left(x_1 - \frac{2x_2}{x_1^2 - x_2^2 - C_1}\right)\mathbf{j}$$

and

$$\mathbf{q}_2 = \left(-x_2 + \frac{2C_2x_2}{\mathrm{e}^{-2x_1x_2} + C_2}\right)\mathbf{i} + \left(-x_1 + \frac{2C_2x_1}{\mathrm{e}^{-2x_1x_2} + C_2}\right)\mathbf{j}.$$

Now we find the functions Φ_1 and Φ_2 which are solutions of the equations $\mathrm{grad}\,\Phi_{1,2} = \mathbf{q}_{1,2}$:

$$\Phi_1 = \ln\left|x_1^2 - x_2^2 - C_1\right| + x_1x_2 + C_3$$

and

$$\Phi_2 = x_1x_2 + \ln|\mathrm{e}^{-2x_1x_2} + C_2| + C_4.$$

Then the corresponding solutions f_1 and f_2 of (34) have the form

$$f_1 = \mathrm{e}^{\Phi_1} = d_1\left(x_1^2 - x_2^2 - C_1\right)\mathrm{e}^{x_1x_2} = d_1(y^2 - x^2 - C_1)\,\mathrm{e}^{xy}$$

and

$$f_2 = \mathrm{e}^{\Phi_2} = d_2(\mathrm{e}^{-x_1x_2} + C_2\,\mathrm{e}^{x_1x_2}) = d_2(\mathrm{e}^{-xy} + C_2\,\mathrm{e}^{xy}) \tag{47}$$

where d_1 and d_2 are arbitrary constants.

Thus starting with a particular solution of (34) we constructed two classes of solutions for the same Schrödinger equation.

Example 32. In order to illustrate the Cauchy integral theorem for the Schrödinger equation let us use the following two particular solutions from the preceding example. Let $f_0(x, y) = \mathrm{e}^{xy}$ and as f_1 we choose the function from (47) when $C_2 = 0$ and $d_2 = 1$, $f_1(x, y) = \mathrm{e}^{-xy}$. Both f_0 and f_1 are solutions of (34) with the same potential u in a whole plane. Thus we can apply theorem 26 and consider Γ being for example a unit circle with centre at the origin. Then

$$\begin{aligned}
&\mathrm{Re}\int_\Gamma \partial_z\left(\frac{f_1}{f_0}\right)\mathrm{d}z + \mathrm{i}\,\mathrm{Im}\int_\Gamma f_0^2\partial_z\left(\frac{f_1}{f_0}\right)\mathrm{d}z\\
&\qquad = 2\,\mathrm{Re}\int_\Gamma(-y+\mathrm{i}x)\,\mathrm{e}^{-2xy}\,\mathrm{d}z + 2\,\mathrm{i}\,\mathrm{Im}\int_\Gamma(-y+\mathrm{i}x)\,\mathrm{d}z\\
&\qquad = 2\,\mathrm{Re}\int_0^{2\pi}\mathrm{i}\,(-\sin\tau + \mathrm{i}\cos\tau)\,\mathrm{e}^{-2\cos\tau\sin\tau}\,\mathrm{d}\tau + 2\,\mathrm{i}\,\mathrm{Im}\int_0^{2\pi}\mathrm{i}\,(-\sin\tau+\mathrm{i}\cos\tau)\,\mathrm{d}\tau\\
&\qquad = -2\int_0^{2\pi}\cos\tau\,\mathrm{e}^{-2\cos\tau\sin\tau}\,\mathrm{d}\tau - 2\,\mathrm{i}\int_0^{2\pi}\sin\tau\,\mathrm{d}\tau.
\end{aligned}$$

It is easy to see that both integrals are equal to zero.

Let us calculate the characteristic coefficients for the pair (F_I, G_I) defined by (46). We have

$$a_{(F_I,G_I)} = 0, \qquad b_{(F_I,G_I)} = -\frac{\partial_z f_0}{f_0},$$

$$A_{(F_I,G_I)} = \frac{\rho_{zz}}{\rho_z} - \frac{\rho_{z\bar{z}}}{\rho_{\bar{z}}}, \qquad B_{(F_I,G_I)} = -\frac{f_0'\rho_z^2}{f_0\rho_{\bar{z}}}.$$

Consider the equation

$$\overset{1}{\phi}_{\bar{z}}\, F_I + \overset{1}{\psi}_{\bar{z}}\, G_I = 0$$

where $\overset{1}{\phi}$ and $\overset{1}{\psi}$ are real-valued functions. It has the form

$$\mathrm{i} f_0^2\,\overset{1}{\phi}_{\bar{z}} - \overset{1}{\psi}_{\bar{z}} = 0 \tag{48}$$

or as a system

$$\overset{1}{\psi}_x + f_0^2\,\overset{1}{\phi}_y = 0 \tag{49}$$

$$\overset{1}{\psi}_y - f_0^2\,\overset{1}{\phi}_x = 0. \tag{50}$$

Comparing with system (39), (40) we note that solutions of that system can be transformed into solutions of system (49), (50) in the following way:

$$\overset{1}{\phi} = \psi \qquad \text{and} \qquad \overset{1}{\psi} = -\phi.$$

Let us calculate F_I^* and G_I^* using definition 17:

$$F_I^* = -\frac{f_0\,\mathrm{e}^S}{\rho_z} \qquad \text{and} \qquad G_I^* = -\frac{\mathrm{i}\mathrm{e}^S}{f_0\rho_z}.$$

Consider the equation

$$\overset{2}{\phi}_{\bar z}\, F_I^* + \overset{2}{\psi}_{\bar z}\, G_I^* = 0$$

where $\overset{2}{\phi}$ and $\overset{2}{\psi}$ are real-valued functions. It has the form

$$\mathrm{i} f_0^2\, \overset{2}{\phi}_{\bar z} - \overset{2}{\psi}_{\bar z} = 0. \tag{51}$$

Observe that it coincides with (48). Thus we obtain that the function $v = \phi F_I + \psi G_I$ is (F_I, G_I)-pseudoanalytic iff the function $W = \phi F_I^* + \psi G_I^*$ is (F_I^*, G_I^*)-pseudoanalytic.

Let us calculate

$$B_{(F_I^*, G_I^*)} = \frac{\partial_{\bar z} f_0}{f_0}.$$

That is the characteristic coefficient b of a successor of (F_I^*, G_I^*) is equal to $-\partial_{\bar z} f_0/f_0 = b_{(F,G)}$. Thus, (F, G) is a successor of (F_I^*, G_I^*). This important observation opens the way to obtain an infinite set of solutions of the original Schrödinger equation (34) if f_0 satisfies condition 30. Namely, we start with a solution of (29), for instance with F. Then its (F_I^*, G_I^*) antiderivative gives us an (F_I^*, G_I^*)-pseudoanalytic function, more precisely the corresponding functions ϕ and ψ such that the function $W = \phi F_I^* + \psi G_I^*$ is (F_I^*, G_I^*)-pseudoanalytic. Then we take these real-valued functions ϕ, ψ and consider the function $v = \phi F_I + \psi G_I$ which is (F_I, G_I)-pseudoanalytic. That is v satisfies (30) and hence the vector $\mathbf{F} = v\mathbf{j}$ with the aid of the chain of actions described in remark 10 can be transformed into a solution of (34). Taking the (F, G) antiderivative of iv we obtain another solution of (29) and can start this cycle again. We can represent schematically this procedure for obtaining an infinite sequence of solutions of (30) and consequently of (34) as the following diagram.

$$\begin{tikzcd}
 & & \text{Solutions of (34)} \\
(F,G) \arrow[dr] & (F_I,G_I) \arrow[l] \arrow[ur] & \\
 & (F_I^*,G_I^*) \arrow[u] &
\end{tikzcd}$$

Let us consider how this procedure works on the following example.

Example 33. Here we use the same u and f_0 as in example 31. Then $F = 1/f_0$. We have that $F_I^* = -\mathrm{e}^{xy}/(y - ix)$, $G_I^* = -\mathrm{i}\mathrm{e}^{-xy}/(y - \mathrm{i}x)$. Consider

$$\begin{aligned} * \int_0^z F\, \mathrm{d}_{(F_I^*, G_I^*)} z &= \mathrm{Re} \int_0^z F G_I\, \mathrm{d}z + \mathrm{i}\,\mathrm{Re} \int_0^z F F_I\, \mathrm{d}z \\ &= -\mathrm{Re} \int_0^1 \mathrm{e}^{-2xyt^2}(yt - \mathrm{i}xt)(x + \mathrm{i}y)\,\mathrm{d}t + \mathrm{i}\,\mathrm{Re} \int_0^1 (xt + \mathrm{i}yt)(x + \mathrm{i}y)\,\mathrm{d}t \\ &= \frac{\mathrm{e}^{-2xy} - 1}{2} + \mathrm{i}\frac{(x^2 - y^2)}{2}. \end{aligned}$$

Thus we have $\phi = (\mathrm{e}^{-2xy} - 1)/2$ and $\psi = (x^2 - y^2)/2$. It is easy to check that in fact $\phi_{\bar{z}} F_I^* + \psi_{\bar{z}} G_I^* = 0$, so the function

$$W = \phi F_I^* + \psi G_I^* = -\frac{1}{2(y - \mathrm{i}x)}(\mathrm{e}^{-xy} - \mathrm{e}^{xy} + \mathrm{i}\mathrm{e}^{-xy}(x^2 - y^2))$$
$$= \frac{\sinh(xy)}{y - \mathrm{i}x} - \frac{\mathrm{i}\mathrm{e}^{-xy}(x^2 - y^2)}{2(y - \mathrm{i}x)}$$

is (F_I^*, G_I^*)-pseudoanalytic. Now we can construct an (F_I, G_I)-pseudoanalytic function as follows:

$$v = \phi F_I + \psi G_I = -\frac{(y - \mathrm{i}x)}{2}(\mathrm{e}^{-xy}(x^2 - y^2) - \mathrm{i}(\mathrm{e}^{-xy} - \mathrm{e}^{xy}))$$
$$= -(y - \mathrm{i}x)\left(\frac{\mathrm{e}^{-xy}(x^2 - y^2)}{2} + \mathrm{i}\sinh(xy)\right).$$

Thus v is a solution of (30) and applying the procedure described above it can be transformed into a solution of (34).

Now multiplying the function v by i we obtain a solution of (32) the (F, G) antiderivative of which gives us a new solution of (29) and the cycle starts again.

Thus starting with a particular solution f_0 of (34) and hence with a particular solution F of (29) we obtain an infinite sequence of solutions of (34).

6. Conclusions

In the first part of this work a generalization of the Euler theorem for the quaternionic Riccati equation was presented. For the Schrödinger equation it implies that given a particular solution, the general solution reduces to a first-order quaternionic differential equation which is equivalent to the static Maxwell system. In the second part we analysed the case of two independent variables. Then the first-order quaternionic equation becomes the Vekua equation describing pseudoanalytic or generalized analytic functions. Consideration of a complete (four-component) quaternionic solution suggests an auxiliary Vekua equation for which we are able to construct the Bers generating pair explicitly. Solutions of the main Vekua equation result to be Bers derivatives of solutions of the auxiliary equation multiplied by i. This fact gives us the possibility of applying some of the power of Bers' theory of pseudoanalytic functions in order to obtain such general results for the Schrödinger equation as analogues of the Cauchy integral theorem and of the Morera theorem.

Moreover, for a quite ample class of potentials we succeed in constructing explicitly a generating pair for the main Vekua equation which implies an interesting cyclic procedure producing an infinite set of exact solutions to the original Schrödinger equation. Among other possible potentials admitting the procedure there are, for instance, all radial potentials.

It is clear that in the present work only some possible applications of the main results were considered. The revealed relation between the Schrödinger equation and the first-order quaternionic equation or in a two-dimensional situation with the Vekua equation can be used for analysis of boundary value problems, for obtaining integral representations of solutions and possibly for other interesting applications.

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